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On the definition of an admitted Lie group for stochastic differential equations with multi-Brownian motion

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Abstract
The definition of an admitted Lie group of transformations for stochastic differential equations has been already presented for equations with one-dimensional Brownian motion. The transformation of the dependent variables involves time as well, and it has been proven that Brownian motion is transformed to Brownian motion. In this paper, we will discuss this concept for stochastic differential equations involving multi-dimensional Brownian motion and present applications to a variety of stochastic differential equations.

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1. Introduction

In general, almost all differential equations are very difficult to solve explicitly. Numerical methods are frequently used with much success for obtaining approximate solutions. However, exact solutions are interesting because with their help, one can analyse the properties of the equations studied. One of the methods used for finding exact solutions of differential equations is group analysis.

A survey of this method can be found in [1, 2]. It involves the study of symmetries of equations, by which one means a local group of transformations mapping a solution of a given system of equations to a solution of the same system. Moreover, symmetries allow one to find new solutions of the system.

In contrast to deterministic differential equations, there have been only a few attempts to apply symmetry techniques to stochastic differential equations. They fall into two groups and are outlined in the following.

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space. It is assumed that the σ-algebra $\mathcal{F}$ is a filtration, that is, $\mathcal{F}$ is generated by a family of σ-algebras $\mathcal{F}_t$ $(t \geqslant 0)$ such that

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F} \quad \forall s \leqslant t, \quad s, t \in I,$$

where $I = [0, T]$, $T \in (0, \infty]$.

Let $\{X(t) = (X_1(t), \ldots, X_n(t))\}_{t\geqslant 0}$ be a stochastic process. The system of Itô equations

$$\mathrm{d}X_i(t, \omega) = f_i(t, X(t, \omega))\,\mathrm{d}t + g_{ik}(t, X(t, \omega))\,\mathrm{d}B_k(t, \omega) \qquad (i = 1, \ldots, n, k = 1, \ldots, r) \tag{1}$$

with initial condition $X(0) = X^{(0)}$ is interpreted in the sense that

$$X_i(t, \omega) = X_i^{(0)}(\omega) + \int_0^t f_i(s, X(s, \omega))\,\mathrm{d}s + \int_0^t g_{ik}(s, X(s, \omega))\,\mathrm{d}B_k(s, \omega), \tag{2}$$

for almost all $\omega \in \Omega$ and for each $t > 0$, where $f_i(t, X)$ is a drift vector, $g_{ik}(t, X)$ is a diffusion matrix and $B_k (k = 1, \ldots, r)$ are one-dimensional Brownian motions, $\int_0^t f(s, X(s))\,\mathrm{d}s$ is a Riemann integral and $\int_0^t g(s, X(s))\,\mathrm{d}B(s)$ is an Itô integral; the repeat index k denotes summation.

For example, in the Black–Scholes model, the price of a risky asset is described by the stochastic differential equation [3]

$$\mathrm{d}X(t) = \mu X(t)\,\mathrm{d}t + \sigma X(t)\,\mathrm{d}B(t), \qquad t \in [0, T] \tag{3}$$

with initial condition $X(0) = X^{(0)}$. That is,

$$X(t) = X^{(0)} + \int_0^t \mu X(s)\,\mathrm{d}s + \int_0^t \sigma X(s)\,\mathrm{d}B(s),$$

for each $t \in [0, T]$, where μ is the mean rate of return, σ is the volatility, B is Brownian motion and T is the time of maturity. The solution of equation (3) with the initial condition $X(0) = X^{(0)}$, called geometric Brownian motion, is

$$X(t, \omega) = X^{(0)}(\omega) \exp\left(\sigma B(s, \omega) + \left(\mu - \tfrac{1}{2}\sigma^2\right)t\right).$$

The first approach [4–7] of applying group analysis to stochastic differential equations deals with fibre-preserving transformations only,

$$\bar{x}_i = \varphi_i(t, x, a), \qquad \bar{t} = H(t, a) \qquad (i = 1, \ldots, n), \tag{4}$$

where a is a parameter of a Lie group of transformations. For ease of notation, we use the symbol x in a Lie group of transformations to denote a transformation of a stochastic process X.

By using Itô's formula, transformation (4) maps (1) into the system

$$\mathrm{d}\overline{X}_i = \overline{f}_i(\bar{t}, \overline{X})\,\mathrm{d}t + \bar{g}_{ik}(\bar{t}, \overline{X})\,\mathrm{d}B_k.$$

Recall that according to Itô's formula [8], the evolution of a scalar function $I(t, x)$ satisfies the condition

$$\mathrm{d}I = \left(I_{,t} + f_j I_{,j} + \tfrac{1}{2} g_{jk} g_{lk} I_{,jl}\right)\mathrm{d}t + I_{,j} g_{jk}\,\mathrm{d}B_k, \tag{5}$$

where the comma denotes differentiation, for example, $I_{,t}$ is the partial derivative of I with respect to t.

The requirement that an infinitesimal transformation maps every solution of (1) to a solution of the same system gives the definition of an admitted Lie group for stochastic differential equations. This approach has been applied to stochastic dynamical systems [4, 5] and to the Fokker–Planck equation [6, 7]. Its weakness is that it can only be applied to fibre-preserving transformations.

In the second approach [9–13], the authors have tried to generalize (4). The second approach deals with symmetry transformations for system (1) involving all the dependent variables in the transformation

$$\overline{x}_i = \varphi_i(t, x, a), \qquad \overline{t} = H(t, x, a) \qquad (i = 1, \ldots, n). \tag{6}$$

The transformation of Brownian motions is defined by the formulae

$$\mathrm{d}\overline{B}_k = \mathrm{d}B_k + \tfrac{1}{2}a\left(\tau_{,t} + f_j\tau_{,j} + \tfrac{1}{2}g_{jm}g_{lm}\tau_{,jl}\right)\mathrm{d}B_k, \qquad (k = 1, \ldots, r), \tag{7}$$

where $\tau(t, x) = \frac{\partial H}{\partial a}(t, x, 0)$ is the coefficient of the infinitesimal generator of the Lie group.

This approach has been applied to scalar second-order stochastic ordinary differential equations [9, 10], to the Hamiltonian–Stratonovich dynamical control system [11] and to the Fokker–Planck equation [11–13]. Unfortunately, there is no strict proof that the transformation of Brownian motion $\overline{B}_k$ satisfies the properties of Brownian motion.

In [14], a new definition of an admitted Lie group of transformations for stochastic differential equations was presented. This new approach gives a correct generalization of approach (4). It includes all dependent as well as independent variables in the transformation. In particular, the transformation of Brownian motion is defined by the transformation of the dependent and independent variables, and there is a strict proof that the transformed Brownian motion satisfies the properties of Brownian motion. This transformation of Brownian motion is a logical generalization of the change of variable formula to the Itô integral in the case where the stochastic process is included in the change. The theory developed in [14] discusses equations with one-dimensional Brownian motion only.

This manuscript extends the discussion in [14] to systems of stochastic differential equations and multi-dimensional Brownian motion, and shows how to construct the determining equations for admitted Lie groups of transformations.

2. Lie group of transformations for a system of stochastic differential equations with one-dimensional Brownian motion

This section is devoted to reviewing the theory developed for one-dimensional Brownian motion in [14], as it applies to systems of equations.

2.1. Lie group of transformations for a stochastic process

Assume that the set of transformations

$$\bar{t} = H(t, x, a), \qquad \bar{x} = \varphi(t, x, a) \tag{8}$$

composes a Lie group. Let $h(t, x) = \frac{\partial H}{\partial a}(t, x, 0)$, $\xi(t, x) = \frac{\partial \varphi}{\partial a}(t, x, 0)$ be the coefficients of the infinitesimal generator

$$h(t, x)\partial_t + \xi(t, x)\partial_x.$$

According to Lie's theorem, the functions $H(t, x, a)$ and $\varphi(t, x, a)$ satisfy the Lie equations

$$\frac{\partial H}{\partial a} = h(H, \varphi), \qquad \frac{\partial \varphi}{\partial a} = \xi(H, \varphi) \tag{9}$$

and the initial conditions for $a = 0$:

$$H = t, \qquad \varphi = x. \tag{10}$$

Since $\frac{\partial H}{\partial t}(t, x, 0) = 1$, then $\frac{\partial H}{\partial t}(t, x, a) > 0$ in a neighbourhood of $a = 0$, where one can find a function $\eta(t, x, a)$ such that

$$\eta^2(t, x, a) = \frac{\partial H}{\partial t}(t, x, a).$$

Using the function $\eta(t,x,a)$, one can define a transformation of a stochastic process $X(t,\omega)$ by

$$\bar{X}(\bar{t},\omega) = \varphi(\alpha(\bar{t}), X(\alpha(\bar{t}),\omega), a), \tag{11}$$

where

$$\beta(t) = \int_0^t \eta^2(s, X(s,\omega), a)\,\mathrm{d}s, \qquad t \geqslant 0,$$

and $\alpha(t)$ is the inverse function of $\beta(t)$. This gives an action of Lie group (8) on the set of stochastic processes. Replacing $\bar{t}$ by $\beta(t)$ in (11), one gets

$$\bar{X}(\beta(t),\omega) = \varphi(t, X(t,\omega), a).$$

In calculations of an admitted Lie group of transformations[1] it is useful to introduce the function

$$\tau(t,x) = \frac{\partial \eta}{\partial a}(t,x,0).$$

Note that the functions $h(t,x)$ and $\tau(t,x)$ are related by the formulae

$$\tau(t,x) = \frac{1}{2}\frac{\partial h}{\partial t}(t,x), \qquad h(t,x) = 2\int_0^t \tau(s,x)\,\mathrm{d}s.$$

Similar to partial differential equations, the functions $\tau(t,x)$ and $\xi(t,x)$ define a Lie group of transformations for stochastic processes. In fact, given $\tau(s,x)$ and $\xi(s,x)$, one sets

$$h(t,x) = 2\int_0^t \tau(s,x)\,\mathrm{d}s.$$

Solving the Lie equations (9) with initial conditions (10), one finds the functions $H(t,x,a)$ and $\varphi(t,x,a)$.

2.2. Determining equations

Let us consider the system of Itô equations

$$X_i(t,\omega) = X_i(0,\omega) + \int_0^t f_i(s, X(s,\omega))\,\mathrm{d}s + \int_0^t g_i(s, X(s,\omega))\,\mathrm{d}B(s), \qquad (i = 1,\ldots,n) \tag{12}$$

where the drift rate f and the volatility g are given adapted stochastic processes and B is one-dimensional Brownian motion.

Definition (see [14]). *A Lie group of transformations (8) is called admitted by the stochastic differential equation (12), if for any solution $X(t,\omega)$ of (12) the functions $\xi(t,x)$ and $\tau(t,x)$ satisfy the system of determining equations*

$$\begin{aligned}
&\xi_{i,t}(t, X(t,\omega)) + f_j\xi_{i,j}(t, X(t,\omega)) + \frac{1}{2}g_j g_k \xi_{i,jk}(t, X(t,\omega)) \\
&\quad - 2f_{i,t}(t, X(t,\omega))\int_0^t \tau(s, X(s,\omega))\,\mathrm{d}s \\
&\quad - f_{i,j}\xi_j(t, X(t,\omega)) - 2f_i\tau(t, X(t,\omega)) = 0, \\
&g_j\xi_{i,j}(t, X(t,\omega)) - 2g_{i,t}(t, X(t,\omega))\int_0^t \tau(s, X(s,\omega))\,\mathrm{d}s - g_i\tau(t, X(t,\omega)) \\
&\quad - g_{i,j}\xi_j(t, X(t,\omega)) = 0 \qquad (i = 1,\ldots,n).
\end{aligned} \tag{13}$$

[1] The proper definition of an admitted Lie group of transformation will be given in the following section.

The determining equations for an admitted Lie group of transformations were constructed in [14] under the assumption that the Lie group of transformations (9) transforms any solution of equation (12) into a solution of the same equation.

3. Admitted Lie group of transformations for a system of stochastic differential equations with multi-dimensional Brownian motion

This section is devoted to constructing determining equations of an admitted Lie group of transformations for stochastic differential equations with multi-dimensional Brownian motion.

The constructions below are similar to the theory developed in [14] for one-dimensional Brownian motion. Let $\eta(t, x, a)$ be a sufficiently many times continuously differentiable function and $\{X(t)\}_{t\geqslant 0}$ a continuous and $\mathcal{F}_t$-adapted stochastic process. Since $\eta^2(t, x, a)$ is continuous, $\eta^2(t, X(t, \omega), a)$ is also an $\mathcal{F}_t$-adapted process. Define

$$\beta(t, \omega, a) = \int_0^t \eta^2(s, X(s, \omega), a)\,\mathrm{d}s, \qquad t \geqslant 0. \tag{14}$$

For brevity, we write $\beta(t)$ instead of $\beta(t, \omega, a)$. The function $\beta(t)$ is called a random variable of the time course with time change rate $\eta^2(t, X(t, \omega), a)$. Note that $\beta(t)$ is an $\mathcal{F}_t$-adapted process. Suppose now that $\eta(t, x, a) \neq 0$ for all (t, x, a). Then for each ω, the map $t \longmapsto \beta(t)$ is strictly increasing. Next define

$$\alpha(t, \omega, a) = \inf_{s\geqslant 0}\{s : \beta(s, \omega, a) > t\}, \tag{15}$$

and for brevity, write $\alpha(t)$ instead of $\alpha(t, \omega, a)$. For each ω, the map $t \longmapsto \alpha(t)$ is nondecreasing and continuous. One easily shows that for almost all ω, and for all $t \geqslant 0$,

$$\beta(\alpha(t)) = t = \alpha(\beta(t)). \tag{16}$$

In [14], it was proven that the processes

$$\bar{B}_k(t) = \int_0^{\alpha(t)} \eta(s, X(s, \omega), a)\,\mathrm{d}B_k(s), \qquad t \geqslant 0, \quad (k = 1, \ldots, r)$$

are standard Brownian motions. Consider

$$\psi(t, \omega) = \bar{X}(\beta(t), \omega),$$

where

$$\bar{X}(t, \omega) = \varphi(\alpha(t), X(\alpha(t), \omega), a)$$

is the transformation of the stochastic process $X(t, \omega)$ given by (11). For almost all ω, there is the relation

$$\psi(t, \omega) = \varphi(t, X(t, \omega), a).$$

According to the time change formula for Itô integrals [15], a nonanticipating functional e with

$$\mathcal{P}\left(\int_0^t e^2\,\mathrm{d}s + \int_0^t \eta^2\,\mathrm{d}s < \infty,\ t \geqslant 0\right) = 1$$

satisfies the formula

$$\int_0^{\alpha(t)} e(s, \omega)\,\mathrm{d}B(s) = \int_0^t e(\alpha(s), \omega)\frac{1}{\eta(\alpha(s), X(\alpha(s), \omega), a)}\,\mathrm{d}\bar{B}(s). \tag{17}$$

Now let the set of transformations (8) compose a Lie group. Assume that $\{X(t)\}_{t\geqslant 0}$ is a stochastic process satisfying the equation

$$X_i(t,\omega) = X_i(0,\omega) + \int_0^t f_i(s, X(s,\omega))\,\mathrm{d}s + \int_0^t g_{ik}(s, X(s,\omega))\,\mathrm{d}B_k(s), \qquad (i=1,\ldots,n,\ k=1,\ldots,r), \tag{18}$$

where the drift vector $f = (f_1,\ldots,f_n)$ and the diffusion matrix $g = (g_{ik})_{n\times r}$ are adapted stochastic processes and $B = (B_1,\ldots,B_r)$ is multi-dimensional Brownian motion. Applying Itô's formula to the function $\psi(t,x) = \varphi(t,x,a)$, one has

$$\psi(t,\omega) = \psi(0,\omega) + \int_0^t \left(\varphi_{,t} + f_j\varphi_{,j} + \frac{1}{2}g_{jk}g_{lk}\varphi_{,jl}\right)(s, X(s,\omega), a)\,\mathrm{d}s + \int_0^t g_{jk}\varphi_{,j}(s, X(s,\omega), a)\,\mathrm{d}B_k(s). \tag{19}$$

Because $X(t,\omega)$ is a solution of (18) and $\varphi_x(t,x,a)$ is a continuous function, $g(t,X(t,\omega))\varphi_x(t,X(t,\omega),a)$ is a continuous process and $g\varphi_x$ is a nonanticipating functional. Applying formula (17) to the last term of equation (19), one obtains

$$\psi_i(t,\omega) = \psi_i(0,\omega) + \int_0^{\beta(t)} \left(\varphi_{i,t} + f_j\varphi_{i,j} + \frac{1}{2}g_{jk}g_{lk}\varphi_{i,jl}\right)(\alpha(s), X(\alpha(s),\omega), a)\alpha_{\bar{t}}(s)\,\mathrm{d}s + \int_0^{\beta(t)} \frac{g_{jk}\varphi_{i,j}}{\eta}(\alpha(s), X(\alpha(s),\omega), a)\,\mathrm{d}\bar{B}_k(s), \qquad (i=1,\ldots,n). \tag{20}$$

Since $\beta(t,\omega,a) = \int_0^t \eta^2(s, X(s,\omega), a)\,\mathrm{d}s$ and $\beta(\alpha(\bar{t})) = \bar{t}$ for almost all ω, then

$$\eta^2(\alpha(\bar{t}), X(\alpha(\bar{t}),\omega), a)\alpha_{\bar{t}}(\bar{t}) = 1.$$

This gives

$$\alpha_{\bar{t}}(s) = \eta^{-2}(\alpha(s), X(\alpha(s),\omega), a). \tag{21}$$

Substitution of $\alpha_{\bar{t}}(s)$ into (20) leads to the equation

$$\psi_i(t,\omega) = \psi_i(0,\omega) + \int_0^{\beta(t)} \left(\frac{\varphi_{i,t} + f_j\varphi_{i,j} + \frac{1}{2}g_{jk}g_{lk}\varphi_{i,jl}}{\eta^2}\right)(\alpha(s), X(\alpha(s),\omega), a)\,\mathrm{d}s + \int_0^{\beta(t)} \frac{g_{jk}\varphi_{i,j}}{\eta}(\alpha(s), X(\alpha(s),\omega), a)\,\mathrm{d}\bar{B}_k(s), \qquad (i=1,\ldots,n). \tag{22}$$

Requiring that transformations (8) map a solution of equation (18) into a solution of the same equation, one obtains

$$\bar{X}_i(\bar{t},\omega) = \bar{X}_i(0,\omega) + \int_0^{\bar{t}} f_i(s, \bar{X}(s,\omega))\,\mathrm{d}s + \int_0^{\bar{t}} g_{ik}(s, \bar{X}(s,\omega))\,\mathrm{d}\bar{B}_k(s).$$
$$(i=1,\ldots,n, k=1,\ldots,r)$$

Substituting $\bar{t} = \beta(t)$ into this equation, one gets

$$\bar{X}_i(\beta(t),\omega) = \bar{X}_i(0,\omega) + \int_0^{\beta(t)} f_i(s, \bar{X}(s,\omega))\,\mathrm{d}s + \int_0^{\beta(t)} g_{ik}(s, \bar{X}(s,\omega))\,\mathrm{d}\bar{B}_k(s), \qquad (i=1,\ldots,n,\ k=1,\ldots,r). \tag{23}$$

Equations (22) and (23) will certainly be equal if the integrands of the two Riemann integrals as well as those of the Itô integrals coincide. Comparing the Riemann and Itô integrals, respectively, one obtains

$$\left(\varphi_{i,t} + f_j\varphi_{i,j} + \tfrac{1}{2}g_{jk}g_{lk}\varphi_{i,jl}\right)(\alpha(t), X(\alpha(t),\omega), a) = f_i(t, \bar{X}(t,\omega))\eta^2(\alpha(t), X(\alpha(t),\omega), a), \tag{24}$$

$$g_{jk}\varphi_{i,j}(\alpha(t), X(\alpha(t),\omega), a) = g_{ik}(t, \bar{X}(t,\omega))\eta(\alpha(t), X(\alpha(t),\omega), a),$$
$$(i = 1,\dots,n,\ k = 1,\dots,r). \tag{25}$$

Since $\bar{X}(\bar{t},\omega) = \varphi(\alpha(\bar{t}), X(\alpha(\bar{t}),\omega), a)$, equations (24) and (25) become

$$\begin{aligned}&\left(\varphi_{i,t} + f_j\varphi_{i,j} + \tfrac{1}{2}g_{jk}g_{lk}\varphi_{i,jl}\right)(\alpha(\bar{t}), X(\alpha(\bar{t}),\omega), a)\\ &\qquad = f_i(\bar{t}, \varphi(\alpha(\bar{t}), X(\alpha(\bar{t}),\omega), a))\eta^2(\alpha(\bar{t}), X(\alpha(\bar{t}),\omega), a),\end{aligned} \tag{26}$$

$$g_{jk}\varphi_{i,j}(\alpha(\bar{t}), X(\alpha(\bar{t}),\omega), a) = g_{ik}(\bar{t}, \varphi(\alpha(\bar{t}), X(\alpha(\bar{t}),\omega), a))\eta(\alpha(\bar{t}), X(\alpha(\bar{t}),\omega), a),$$
$$(i = 1,\dots,n,\ k = 1,\dots,r). \tag{27}$$

Substituting $\bar{t} = \beta(t)$ into equations (26) and (27), the two equations can be rewritten as

$$\left(\varphi_{i,t} + f_j\varphi_{i,j} + \tfrac{1}{2}g_{jk}g_{lk}\varphi_{i,jl}\right)(t, X(t,\omega), a) = f_i(\beta(t), \varphi(t, X(t,\omega), a))\eta^2(t, X(t,\omega), a), \tag{28}$$

$$g_{jk}\varphi_{i,j}(t, X(t,\omega), a) = g_{ik}(\beta(t), \varphi(t, X(t,\omega), a))\eta(t, X(t,\omega), a),$$
$$(i = 1,\dots,n,\ k = 1,\dots,r). \tag{29}$$

Differentiating equations (28) and (29) with respect to the parameter a, one obtains the equations

$$\begin{aligned}&\left(\varphi_{i,ta} + f_j\varphi_{i,ja} + \frac{1}{2}g_{jk}g_{lk}\varphi_{i,jla}\right)(t, X(t,\omega), a)\\ &\qquad = \left(\eta^2\left(f_{i,t}\frac{\partial\beta}{\partial a} + f_{i,j}\varphi_{j,a}\right) + 2f_i\eta\eta_a\right)(t, X(t,\omega), a),\end{aligned} \tag{30}$$

$$g_{jk}\varphi_{i,ja}(t, X(t,\omega), a) = \left(g_{ik}\eta_a + \eta\left(g_{ik,t}\frac{\partial\beta}{\partial a} + g_{ik,j}\varphi_{j,a}\right)\right)(t, X(t,\omega), a),$$
$$(i = 1,\dots,n,\ k = 1,\dots,r). \tag{31}$$

Substituting $a = 0$ into equations (30) and (31) and using (10), one has

$$\begin{aligned}&\left(\frac{\partial\varphi_i}{\partial a}\Big|_{a=0}\right)_t + f_j\left(\frac{\partial\varphi_i}{\partial a}\Big|_{a=0}\right)_{,j} + \frac{1}{2}g_{jk}g_{lk}\left(\frac{\partial\varphi_i}{\partial a}\Big|_{a=0}\right)_{,jl}\\ &\qquad = f_{i,t}\frac{\partial\beta}{\partial a}\Big|_{a=0} + f_{i,j}\frac{\partial\varphi_j}{\partial a}\Big|_{a=0} + 2f_i\frac{\partial\eta}{\partial a}\Big|_{a=0},\end{aligned} \tag{32}$$

$$g_{jk}\left(\frac{\partial\varphi_i}{\partial a}\Big|_{a=0}\right)_{,j} = g_{ik,t}\frac{\partial\beta}{\partial a}\Big|_{a=0} + g_{ik}\frac{\partial\eta}{\partial a}\Big|_{a=0} + g_{ik,j}\frac{\partial\varphi_j}{\partial a}\Big|_{a=0},$$
$$(i = 1,\dots,n,\ k = 1,\dots,r). \tag{33}$$

Since $\beta(t,\omega,a) = \int_0^t \eta^2(s, X(s,\omega), a)\,\mathrm{d}s$ for all $t \geqslant 0$, differentiating this with respect to a, one finds

$$\frac{\partial\beta}{\partial a}\Big|_{a=0} = 2\int_0^t \frac{\partial\eta}{\partial a}\Big|_{a=0}\mathrm{d}s.$$

Substituting $\frac{\partial\beta}{\partial a}\big|_{a=0}$ into equations (32) and (33), one arrives at the following equations

$$\begin{aligned}
&\xi_{i,t}(t, X(t,\omega)) + f_j\xi_{i,j}(t, X(t,\omega)) + \frac{1}{2}g_{jk}g_{lk}\xi_{i,jl}(t, X(t,\omega)) \\
&\qquad - 2f_{i,t}(t, X(t,\omega))\int_0^t \tau(s, X(s,\omega))\,\mathrm{d}s \\
&\qquad - f_{i,j}\xi_j(t, X(t,\omega)) - 2f_i\tau(t, X(t,\omega)) = 0, \\
&g_{jk}\xi_{i,j}(t, X(t,\omega)) - 2g_{ik,t}(t, X(t,\omega))\int_0^t \tau(s, X(s,\omega))\,\mathrm{d}s \\
&\qquad - g_{ik}\tau(t, X(t,\omega)) - g_{ik,j}\xi_j(t, X(t,\omega)) = 0, \\
&\qquad (i = 1,\ldots,n,\ k = 1,\ldots,r).
\end{aligned} \tag{34}$$

Equations (34) are integro-differential equations for the functions $\tau(t,x)$ and $\xi(t,x)$. These equations have to be satisfied for any solution $X(t,\omega)$ of stochastic differential equation (18). Thus, one can define an admitted Lie group by using the determining equations (34).

Definition. *A Lie group of transformations (8) is called admitted by the stochastic differential equation (18), if for any solution $X(t,\omega)$ of (18), the functions $\xi(t,x)$ and $\tau(t,x)$ satisfy the determining equations (34).*

Assume that one has found the functions $\tau(t,x)$ and $\xi(t,x)$ which are solutions of the determining equations (34). Then the Lie group of transformations (8) is recovered by solving the Lie equations

$$\frac{\partial H}{\partial a}(t,x,a) = h(H,\varphi), \qquad \frac{\partial \varphi}{\partial a}(t,x,a) = \xi(H,\varphi),$$

with the initial conditions

$$H(t,x,0) = t, \qquad \varphi(t,x,0) = x,$$

where $h(t,x) = 2\int_0^t \tau(s,x)\,\mathrm{d}s$, and

$$\eta^2 = \frac{\partial H}{\partial t}.$$

4. Stochastic differential equations with one-dimensional Brownian motion

In the following, we present examples of systems of two equations involving a single Brownian motion. For convenience of notation, we will use the symbols X and Y instead of X_1 and X_2.

4.1. Graph of Brownian motion

Consider the system of equations [8]

$$\mathrm{d}X(t) = \mathrm{d}t, \qquad \mathrm{d}Y(t) = \mathrm{d}B(t). \tag{35}$$

The solution of equations (35) with the initial condition $(X(0), Y(0)) = (t_0, y_0)$ may be regarded as the graph of Brownian motion. For equations (35) the corresponding functions of equations (12) are $f_1 = 1$, $f_2 = 0$, $g_1 = 0$ and $g_2 = 1$. The system of determining equations for (35) becomes

$$\xi_{1,t} + \xi_{1,x} + \tfrac{1}{2}\xi_{1,yy} - 2\tau = 0, \qquad \xi_{2,t} + \xi_{2,x} + \tfrac{1}{2}\xi_{2,yy} = 0, \qquad \xi_{1,y} = 0, \qquad \xi_{2,y} - \tau = 0. \tag{36}$$

The general solution of determining equations (36) is

$$\xi_1 = 2xF_1 + F_3, \qquad \xi_2 = yF_1 + F_2, \qquad \tau = F_1, \tag{37}$$

where $F_1 = F_1(t-x)$, $F_2 = F_2(t-x)$ and $F_3 = F_3(t-x)$. A basis of generators corresponding to (37) is

$$F_2\partial_y, \quad F_3\partial_x, \quad F_1(2x\partial_x + y\partial_y) + h\partial_t,$$

where $h = 2\int_0^t F_1(s-x)\,\mathrm{d}s$.

Note that if the first equation of (35) is considered as an ordinary differential equation (i.e., the function $X(t,\omega)$ does not depend on ω), then the functions F_1, F_2 and F_3 are constants.

4.2. Black and Scholes market

Consider the system of equations [8]

$$\mathrm{d}X(t) = \rho X(t)\,\mathrm{d}t, \qquad \mathrm{d}Y(t) = \mu Y(t)\,\mathrm{d}t + \sigma Y(t)\,\mathrm{d}B(t), \tag{38}$$

where ρ, μ and σ are nonzero constants. The system of equations (38) with the initial condition $X(0) = 1, Y(0) = y > 0$ is called the Black and Scholes market. For equations (38) the corresponding functions of equations (12) are $f_1 = \rho x$, $f_2 = \mu y$, $g_1 = 0$ and $g_2 = \sigma y$. The system of determining equations for (38) becomes

$$\begin{aligned}
&\xi_{1,t} + \rho x\xi_{1,x} + \mu y\xi_{1,y} + \tfrac{1}{2}\sigma^2 y^2\xi_{1,yy} - 2\rho x\tau - \rho\xi_1 = 0,\\
&\xi_{2,t} + \rho x\xi_{2,x} + \mu y\xi_{2,y} + \tfrac{1}{2}\sigma^2 y^2\xi_{2,yy} - 2\mu y\tau - \mu\xi_2 = 0,\\
&y\xi_{1,y} = 0, \quad y\xi_{2,y} - y\tau - \xi_2 = 0.
\end{aligned} \tag{39}$$

The general solution of determining equations (39) is

$$\xi_1 = 2x\ln x F_1 + xF_3, \qquad \xi_2 = (y\ln y + \gamma y\ln x)F_1 + yF_2, \qquad \tau = F_1, \tag{40}$$

where $F_1 = F_1\big(t - \frac{\ln x}{\rho}\big)$, $F_2 = F_2\big(t - \frac{\ln x}{\rho}\big)$, $F_3 = F_3\big(t - \frac{\ln x}{\rho}\big)$ and $\gamma = \frac{1}{\rho}\big(\mu - \frac{1}{2}\sigma^2\big)$. A basis of generators corresponding to (40) is

$$yF_2\partial_y, \quad xF_3\partial_x, \quad F_1(2x\ln x\partial_x + (y\ln y + \gamma y\ln x)\partial_y) + h\partial_t,$$

where $h = 2\int_0^t F_1\big(s - \frac{\ln x}{\rho}\big)\,\mathrm{d}s$.

4.3. Nonlinear Itô system

Consider the system of equations [8]

$$\mathrm{d}X(t) = \mathrm{d}t, \qquad \mathrm{d}Y(t) = Y(t)\,\mathrm{d}t + \mathrm{e}^{X(t)}\,\mathrm{d}B(t). \tag{41}$$

For equations (41), the corresponding functions of equations (12) are $f_1 = 1$, $f_2 = y$, $g_{11} = 0$ and $g_2 = \mathrm{e}^x$. The system of determining equations for equations (41) becomes

$$\begin{aligned}
&\xi_{1,t} + \xi_{1,x} + y\xi_{1,y} + \tfrac{1}{2}\,\mathrm{e}^x\xi_{1,yy} - 2\tau = 0,\\
&\xi_{2,t} + \xi_{2,x} + y\xi_{2,y} + \tfrac{1}{2}\,\mathrm{e}^x\xi_{2,yy} - 2y\tau - \xi_2 = 0,\\
&\xi_{1,y} = 0, \qquad \xi_{2,y} - \tau - \xi_1 = 0.
\end{aligned} \tag{42}$$

The general solution of determining equations (42) is

$$\xi_1 = F_2 x + F_3, \qquad \xi_2 = F_1\,\mathrm{e}^x + F_2\big(\tfrac{1}{2} + x\big)y + F_3 y, \qquad \tau = \tfrac{1}{2}F_2, \tag{43}$$

where $F_1 = F_1(t-x)$, $F_2 = F_2(t-x)$ and $F_3 = F_3(t-x)$. A basis of generators corresponding to (43) is

$$F_1\,\mathrm{e}^x\partial_y, \quad F_3(\partial_x + y\partial_y), \quad F_2\big(x\partial_x + \big(\tfrac{1}{2} + x\big)y\partial_y\big) + h\partial_t,$$

where $h = 2\int_0^t F_2(s-x)\,\mathrm{d}s$.

Note that if the first equation of (41) is considered as an ordinary differential equation (i.e., the function $X(t,\omega)$ does not depend on ω), then the functions F_1, F_2 and F_3 are constant.

4.4. Mean-reverting Ornstein–Uhlenbeck process

Consider the system of equations [8]

$$\mathrm{d}X(t) = \rho X(t)\,\mathrm{d}t, \qquad \mathrm{d}Y(t) = (m - Y(t))\,\mathrm{d}t + \sigma\,\mathrm{d}B(t), \tag{44}$$

where $\rho > 0, m > 0$ and $\sigma \neq 0$ are constants. For equations (44), the functions in equation (12) are $f_1 = \rho x$, $f_2 = m - y$, $g_1 = 0$ and $g_2 = \sigma$. The system of determining equations for (44) becomes

$$\begin{aligned}
&\xi_{1,t} + \rho x \xi_{1,x} + (m-y)\xi_{1,y} + \tfrac{1}{2}\sigma^2 \xi_{1,yy} - 2\rho x\tau - \rho\xi_1 = 0,\\
&\xi_{2,t} + \rho x \xi_{2,x} + (m-y)\xi_{2,y} + \tfrac{1}{2}\sigma^2 \xi_{2,yy} - 2(m-y)\tau + \xi_2 = 0,\\
&\xi_{1,y} = 0, \qquad \xi_{2,y} - \tau = 0.
\end{aligned} \tag{45}$$

The general solution of determining equations (45) is

$$\xi_1 = \frac{2}{\gamma} x^{\gamma+1} F_1 + x F_3, \qquad \xi_2 = (x^\gamma y - m x^\gamma) F_1 + x^{\frac{\gamma}{2}} F_2, \qquad \tau = x^\gamma F_1, \tag{46}$$

where $F_1 = F_1\big(t - \frac{\ln x}{\rho}\big)$, $F_2 = F_2\big(t - \frac{\ln x}{\rho}\big)$ and $F_3 = F_3\big(t - \frac{\ln x}{\rho}\big)$ and $\gamma = -\frac{2}{\rho}$. A basis of generators corresponding to (46) is

$$x^{\frac{\gamma}{2}-1} F_2 \partial_y, \qquad x F_3 \partial_x, \qquad F_1\left(\frac{2}{\gamma} x^{\gamma+1}\partial_x + (x^\gamma y - m x^\gamma)\partial_y\right) + h\partial_t,$$

where $h = 2\int_0^t x^\gamma F_1\big(s - \frac{\ln x}{\rho}\big)\,\mathrm{d}s$.

Let us construct the Lie group of transformations corresponding to the third generator for the particular case defined by the assumption $F_1 = k$, where k is constant. In this case the generator becomes

$$\frac{2}{\gamma} x^{\gamma+1}\partial_x + (x^\gamma y - m x^\gamma)\partial_y + 2x^\gamma t\partial_t.$$

For finding the Lie group of transformations corresponding to this generator, one has to solve the Lie equations

$$\frac{\partial H}{\partial a} = 2\varphi_2^\gamma H, \qquad \frac{\partial \varphi_1}{\partial a} = \frac{2}{\gamma}\varphi_1^{\gamma+1}, \qquad \frac{\partial \varphi_2}{\partial a} = \varphi_1^\gamma \varphi_2 - m\varphi_1^\gamma,$$

with the initial conditions at $a = 0$:

$$H = t, \qquad \varphi_1 = x, \qquad \varphi_2 = y.$$

The solution of this Cauchy problem gives the transformations of the independent variable t and the dependent variables x and y,

$$\begin{aligned}
&\bar{t} = H = t x^{-\gamma}(x^{-\gamma} - 2a)^{-1}, \qquad \bar{x} = \varphi_1 = (x^{-\gamma} - 2a)^{-\frac{1}{\gamma}},\\
&\bar{y} = \varphi_2 = (y - m)x^{-\frac{\gamma}{2}}(x^{-\gamma} - 2a)^{-\frac{1}{2}} + m.
\end{aligned} \tag{47}$$

Hence $\eta^2 = H_t = x^{-\gamma}(x^{-\gamma} - 2a)^{-1}$.

Let us show that the Lie group of transformation (47) transforms a solution of equations (44) into a solution of the same equations. Assume that $(X(t), Y(t))$ is a solution of equations (44). As was proven, the Brownian motion $B(t)$ is transformed to the Brownian motion

$$\bar{B}(t) = \int_0^{\alpha(t)} X^{-\frac{\gamma}{2}}(s)(X^{-\gamma}(s) - 2a)^{-\frac{1}{2}}\,\mathrm{d}B(s), \qquad t \geqslant 0, \tag{48}$$

where

$$\beta(t) = \int_0^t X^{-\gamma}(s)(X^{-\gamma}(s) - 2a)^{-1}\,\mathrm{d}s, \qquad \alpha(t) = \inf_{s\geqslant 0}\{s : \beta(s) > t\}, \quad t \geqslant 0.$$

Applying Itô's formula to the functions $\varphi_1(t, x, y, a) = (x^{-\gamma} - 2a)^{-\frac{1}{\gamma}}$ and $\varphi_2(t, x, y, a) = (y - m)x^{-\frac{\gamma}{2}}(x^{-\gamma} - 2a)^{-\frac{1}{2}} + m$, one has

$$\begin{aligned}\varphi_1(t, X(t,\omega), Y(t,\omega), a) &= \varphi_1(0, X(0,\omega), Y(0,\omega), a)\\ &\quad + \int_0^t \rho X^{-\gamma-1}(s)(X^{-\gamma}(s) - 2a)^{-\frac{1}{\gamma}-1}\,\mathrm{d}s,\\ \varphi_2(t, X(t,\omega), Y(t,\omega), a) &= \varphi_2(0, X(0,\omega), Y(0,\omega), a)\\ &\quad + \int_0^t (m - Y(s))X^{-\frac{\gamma}{2}}(s)X^{-\gamma}(s)(X(s)^{-\gamma} - 2a)^{-\frac{3}{2}}\,\mathrm{d}s\\ &\quad + \int_0^t \sigma X^{-\frac{\gamma}{2}}(s)(X(s)^{-\gamma} - 2a)^{-\frac{1}{2}}\,\mathrm{d}B(s).\end{aligned} \tag{49}$$

By virtue of (21)

$$\frac{\partial\alpha}{\partial\bar{t}}(\bar{t}) = X^{\gamma}(s)(X^{-\gamma}(s) - 2a).$$

Changing the variable $s = \alpha(\bar{s})$ in the Riemann integrals in (49), they become

$$\int_0^t \rho X^{-\gamma-1}(s)(X^{-\gamma}(s) - 2a)^{-\frac{1}{\gamma}-1}\,\mathrm{d}s = \int_0^{\beta(t)} \rho X^{-\gamma}(\alpha(s)) - 2a)^{-\frac{1}{\gamma}}\,\mathrm{d}\bar{s},$$

$$\begin{aligned}&\int_0^t (m - Y(s))X^{-\frac{\gamma}{2}}(s)X^{-\gamma}(s)(X(s)^{-\gamma} - 2a)^{-\frac{3}{2}}\,\mathrm{d}s\\ &\qquad = \int_0^t (m - Y(\alpha(s)))X^{-\frac{\gamma}{2}}(\alpha(s))(X(\alpha(s))^{-\gamma} - 2a)^{-\frac{1}{2}}\,\mathrm{d}\bar{s}\\ &\qquad = \int_0^t (m - (Y(\alpha(s) - m))X^{-\frac{\gamma}{2}}(\alpha(s))(X(\alpha(s))^{-\gamma} - 2a)^{-\frac{1}{2}} - m)\,\mathrm{d}\bar{s}.\end{aligned}$$

Because of the transformation of the Brownian motion (48), the Itô integral in (49) becomes

$$\int_0^t \sigma X^{-\frac{\gamma}{2}}(s)(X(s)^{-\gamma} - 2a)^{-\frac{1}{2}}\,\mathrm{d}B(s) = \int_0^{\beta(t)} \sigma\,\mathrm{d}\bar{B}(\bar{s}).$$

Since $(Y(\alpha(\bar{t}),\omega) - m)X^{-\frac{\gamma}{2}}(\alpha(\bar{t}),\omega)(X^{-\gamma}(\alpha(\bar{t}),\omega) - 2a)^{-\frac{1}{2}} + m = \bar{Y}(\bar{t},\omega)$ and $(X^{-\gamma}(\alpha(\bar{t}),\omega) - 2a)^{-\frac{1}{\gamma}} = \bar{X}(\bar{t},\omega)$, one gets

$$\begin{aligned}\varphi_1(t, X(t,\omega), Y(t,\omega), a) &= \varphi_1(0, X(0,\omega), Y(0,\omega), a) + \int_0^{\beta(t)} \rho\bar{X}(s,\omega)\,\mathrm{d}s,\\ \varphi_2(t, X(t,\omega), Y(t,\omega), a) &= \varphi_2(0, X(0,\omega), Y(0,\omega), a)\\ &\quad + \int_0^{\beta(t)} (m - \bar{Y}(s,\omega))\,\mathrm{d}s + \int_0^{\beta(t)} \sigma\,\mathrm{d}\bar{B}(s).\end{aligned}$$

Because $\varphi_1(t, X(t,\omega), Y(t,\omega), a) = \bar{X}(\beta(t),\omega)$, and $\varphi_2(t, X(t,\omega), Y(t,\omega), a) = \bar{Y}(\beta(t),\omega)$, one has

$$\bar{X}(\beta(t),\omega) = \bar{X}(0,\omega) + \int_0^{\beta(t)} \rho\bar{X}(s,\omega)\,\mathrm{d}s,$$

$$\bar{Y}(\beta(t),\omega) = \bar{Y}(0,\omega) + \int_0^{\beta(t)} (m - \bar{Y}(s,\omega))\,\mathrm{d}s + \int_0^{\beta(t)} \sigma\,\mathrm{d}\bar{B}(s).$$

This confirms that the Lie group of transformations (47) transforms any solution of system (44) into a solution of the same system.

5. Stochastic differential equations with multi-dimensional Brownian motion

5.1. System of location and motion

Let μ, σ_1 and σ_2 be nonzero constants. Consider the system of equations [16]

$$\mathrm{d}X(t) = Y(t)\,\mathrm{d}t + \sigma_1\,\mathrm{d}B_1(t), \qquad \mathrm{d}Y(t) = -\mu Y(t)\,\mathrm{d}t + \sigma_2\,\mathrm{d}B_2(t). \tag{50}$$

For equations (50), the corresponding functions of equations (34) are $f_1 = y, f_2 = -\mu y, g_{11} = \sigma_1, g_{12} = 0, g_{21} = 0$ and $g_{22} = \sigma_2$. The system of determining equations for (50) becomes

$$\begin{aligned}
&\xi_{1,t} + y\xi_{1,x} - \mu y\xi_{1,y} + \tfrac{1}{2}\sigma_1^2\xi_{1,xx} + \tfrac{1}{2}\sigma_2^2\xi_{1,yy} - 2y\tau - \xi_2 = 0,\\
&\xi_{2,t} + y\xi_{2,x} - \mu y\xi_{2,y} + \tfrac{1}{2}\sigma_1^2\xi_{2,xx} + \tfrac{1}{2}\sigma_2^2\xi_{2,yy} + 2\mu y\tau + \mu\xi_2 = 0,\\
&\xi_{1,x} - \tau = 0, \qquad \xi_{2,y} - \tau = 0, \qquad \xi_{1,y} = 0, \qquad \xi_{2,x} = 0.
\end{aligned} \tag{51}$$

The general solution of determining equations (51) is

$$\xi_1 = -C_1\,\mathrm{e}^{-\mu t} + C_2, \qquad \xi_2 = C_1\mu\,\mathrm{e}^{-\mu t}, \qquad \tau = 0. \tag{52}$$

Hence $h = 0$. Thus, a basis of generators corresponding to (52) is

$$\mathrm{e}^{-\mu t}(\partial_x - \mu\partial_y), \quad \partial_x.$$

5.2. Vibrating string model

Consider the system of equations [8]

$$\mathrm{d}X(t) = Y(t)\,\mathrm{d}t + \mu\mathrm{d}B_1(t), \qquad \mathrm{d}Y(t) = -X(t)\,\mathrm{d}t + \sigma\,\mathrm{d}B_2(t), \tag{53}$$

where μ and σ are nonzero constants. The system of equations (53) is a model for a vibrating string subject to a stochastic force. For equations (53), the corresponding functions of equations (34) are $f_1 = y$, $f_2 = -x$, $g_{11} = \mu$, $g_{12} = 0$, $g_{21} = 0$ and $g_{22} = \sigma$. The system of determining equations for (53) becomes

$$\begin{aligned}
&\xi_{1,t} + y\xi_{1,x} - x\xi_{1,y} + \tfrac{1}{2}\mu^2\xi_{1,xx} + \tfrac{1}{2}\sigma^2\xi_{1,yy} - 2y\tau - \xi_2 = 0,\\
&\xi_{2,t} + y\xi_{2,x} - x\xi_{2,y} + \tfrac{1}{2}\mu^2\xi_{2,xx} + \tfrac{1}{2}\sigma^2\xi_{2,yy} + 2x\tau + \xi_1 = 0,\\
&\xi_{1,x} - \tau = 0, \qquad \xi_{2,y} - \tau = 0, \qquad \xi_{1,y} = 0, \qquad \xi_{2,x} = 0.
\end{aligned} \tag{54}$$

The general solution of determining equations (54) is

$$\xi_1 = C_1\sin t + C_2\cos t, \qquad \xi_2 = C_1\cos t - C_2\sin t, \qquad \tau = 0. \tag{55}$$

Hence $h = 0$. Thus, a basis of generators corresponding to (55) is

$$\sin t\,\partial_x + \cos t\,\partial_y, \qquad \cos t\,\partial_x - \sin t\,\partial_y.$$

5.3. Nonlinear Itô system

Let μ_1 and μ_2 be constants. Consider the system of equations [6]

$$\mathrm{d}X(t) = \frac{\mu_1}{X(t)}\,\mathrm{d}t + \mathrm{d}B_1(t), \qquad \mathrm{d}Y(t) = \mu_2\mathrm{d}t + \mathrm{d}B_2(t). \tag{56}$$

The associated Fokker–Planck equation is

$$u_t = \frac{1}{2}(u_{xx} + u_{yy}) + \frac{\mu_1}{x^2}u - \frac{\mu_1}{x}u_x - \mu_2 u_y.$$

For equations (56), the corresponding functions of equations (34) are $f_1 = \frac{\mu_1}{x}$, $f_2 = \mu_2$, $g_{11} = 1$, $g_{12} = 0$, $g_{21} = 0$ and $g_{22} = 1$. The system of determining equations for (56) becomes

$$\begin{aligned}
&\xi_{1,t} + \frac{\mu_1}{x}\xi_{1,x} + \mu_2\xi_{1,y} + \tfrac{1}{2}\xi_{1,xx} + \tfrac{1}{2}\xi_{1,yy} - 2\frac{\mu_1}{x}\tau + \frac{\mu_1}{x^2}\xi_1 = 0,\\
&\xi_{2,t} + \frac{\mu_1}{x}\xi_{2,x} + \mu_2\xi_{2,y} + \tfrac{1}{2}\xi_{2,xx} + \tfrac{1}{2}\xi_{2,yy} - 2\mu_2\tau = 0,\\
&\xi_{1,x} - \tau = 0, \qquad \xi_{2,y} - \tau = 0, \qquad \xi_{1,y} = 0, \qquad \xi_{2,x} = 0.
\end{aligned} \tag{57}$$

The general solution of determining equations (57) is

$$\xi_1 = C_1 x, \qquad \xi_2 = C_1(y + \mu_2 t) + C_2, \qquad \tau = C_1, \tag{58}$$

and $h = 2C_1 t$. Thus, a basis of generators corresponding to (58) is

$$\partial_y, \quad x\partial_x + (y + \mu_2 t)\partial_y + 2t\partial_t.$$

5.4. *Ornstein–Uhlenbeck process*

Consider the system of equations [6]

$$\mathrm{d}X(t) = -X(t)\,\mathrm{d}t, \qquad \mathrm{d}Y(t) = -Y(t)\,\mathrm{d}t + \mathrm{d}B_1(t) + \mathrm{d}B_2(t). \tag{59}$$

This system represents an Ornstein–Uhlenbeck process and its corresponding Fokker–Planck equation is

$$u_t + \tfrac{1}{2}u_{yy} - xu_x - yu_y - 2u = 0.$$

For system of equations (59), the corresponding functions of equations (34) are $f_1 = -x$, $f_2 = -y$, $g_{11} = 0$, $g_{12} = 0$, $g_{21} = 1$ and $g_{22} = 1$. The system of determining equations for (59) becomes

$$\begin{aligned}
&\xi_{1,t} - x\xi_{1,x} - y\xi_{1,y} + \xi_{1,yy} + 2x\tau + \xi_1 = 0,\\
&\xi_{2,t} - x\xi_{2,x} - y\xi_{2,y} + \xi_{2,yy} + 2y\tau + \xi_2 = 0,\\
&\xi_{1,y} = 0, \qquad \xi_{2,y} - \tau = 0.
\end{aligned} \tag{60}$$

The general solution of determining equations (60) is

$$\xi_1 = x^3 F_1 + xF_3, \qquad \xi_2 = x^2 y F_1 + xF_2, \qquad \tau = x^2 F_1, \tag{61}$$

where $F_1 = F_1(x\,\mathrm{e}^t)$, $F_2 = F_2(x\,\mathrm{e}^t)$ and $F_3 = F_3(x\,\mathrm{e}^t)$. A basis of generators corresponding to (61) is

$$xF_2\partial_y, \quad xF_3\partial_x, \quad F_1 x^2(x\partial_x + y\partial_y) + x^2 h\partial_t,$$

where $h = 2\int_0^t F_1(x\,\mathrm{e}^s)\,\mathrm{d}s$.

Let us construct the Lie group of transformations corresponding to the third generator for the particular case defined by the assumption $F_1 = k$, where k is constant. In this case the generator becomes

$$x^3\partial_x + yx^2\partial_y + 2x^2 t\partial_t.$$

For finding the Lie group of transformations corresponding to this generator, one has to solve the Lie equations

$$\frac{\partial H}{\partial a} = 2\varphi_1^2 H, \qquad \frac{\partial \varphi_1}{\partial a} = \varphi_1^3, \qquad \frac{\partial \varphi_2}{\partial a} = \varphi_2\varphi_1^2,$$

with the initial conditions for $a = 0$:

$$H = t, \quad \varphi_1 = x, \quad \varphi_2 = y.$$

The solution of this Cauchy problem gives the transformations of the independent variable t and the dependent variables x and y,

$$\bar{t} = H = t(1-2ax^2)^{-1}, \qquad \bar{x} = \varphi_1 = x(1-2ax^2)^{-\frac{1}{2}}, \qquad \bar{y} = \varphi_2 = y(1-2ax^2)^{-\frac{1}{2}}. \tag{62}$$

Hence $\eta^2 = (1-2ax^2)^{-1}$.

Let us show that the Lie group of transformations (62) transforms a solution of equations (59) into a solution of the same equations. Assume that $(X(t), Y(t))$ is a solution of equations (59). It was proven that the Brownian motions $B_1(t)$ and $B_2(t)$ are transformed to the Brownian motions

$$\begin{aligned} \bar{B}_1(t) &= \int_0^{\alpha(t)} (1-2aX^2(s))^{-\frac{1}{2}}\,\mathrm{d}B_1(s), \\ \bar{B}_2(t) &= \int_0^{\alpha(t)} (1-2aX^2(s))^{-\frac{1}{2}}\,\mathrm{d}B_2(s), \qquad t \geqslant 0, \end{aligned} \tag{63}$$

where

$$\beta(t) = \int_0^t (1-2aX^2(s))^{-1}\mathrm{d}s, \qquad \alpha(t) = \inf_{s\geqslant 0}\{s : \beta(s) > t\}, \quad t \geqslant 0.$$

Applying Itô's formula to the functions $\varphi_1(t,x,y,a) = x(1-2ax^2)^{-\frac{1}{2}}$ and $\varphi_2(t,x,y,a) = y(1-2ax^2)^{-\frac{1}{2}}$, one has

$$\begin{aligned} \varphi_1(t, X(t,\omega), Y(t,\omega), a) &= \varphi_1(0, X(0,\omega), Y(0,\omega), a) - \int_0^t X(s)(1-2aX^2(s))^{-\frac{3}{2}}\,\mathrm{d}s \\ \varphi_2(t, X(t,\omega), Y(t,\omega), a) &= \varphi_2(0, X(0,\omega), Y(0,\omega), a) - \int_0^t Y(s,\omega)(1-2aX^2(s))^{-\frac{3}{2}}\,\mathrm{d}s \\ &+ \int_0^t (1-2aX^2(s))^{-\frac{1}{2}}\,\mathrm{d}B_1(s) + \int_0^t (1-2aX^2(s))^{-\frac{1}{2}}\,\mathrm{d}B_2(s). \end{aligned} \tag{64}$$

By virtue of (21)

$$\frac{\partial\alpha}{\partial\bar{t}}(\bar{t}) = (1-2aX^2(s)).$$

Changing the variable $s = \alpha(\bar{s})$ in the Riemann integrals in (64), they become

$$\int_0^t X(s)(1-2aX^2(s))^{-\frac{3}{2}}\,\mathrm{d}s = \int_0^{\beta(t)} X(\alpha(\bar{s}))(1-2aX^2(\alpha(\bar{s})))^{-\frac{1}{2}}\,\mathrm{d}\bar{s},$$

$$\int_0^t Y(s)(1-2aX^2(s))^{-\frac{3}{2}}\,\mathrm{d}s = \int_0^{\beta(t)} Y(\alpha(\bar{s}))(1-2aX^2(\alpha(\bar{s})))^{-\frac{1}{2}}\,\mathrm{d}\bar{s}.$$

Because of the transformation of the Brownian motions (63), the Itô integrals in (64) become

$$\int_0^t (1-2aX^2(s))^{-\frac{1}{2}}\,\mathrm{d}B_1(s) = \int_0^{\beta(t)} \mathrm{d}\bar{B}_1(\bar{s}),$$

$$\int_0^t (1-2aX^2(s))^{-\frac{1}{2}}\,\mathrm{d}B_2(s) = \int_0^{\beta(t)} \mathrm{d}\bar{B}_2(\bar{s}).$$

Since $X(\alpha(\bar{t}))(1-2aX^2(\alpha(\bar{t})))^{-\frac{1}{2}} = \bar{X}(\bar{t},\omega)$ and $Y(\alpha(\bar{t}))(1-2aX^2(\alpha(\bar{t})))^{-\frac{1}{2}} = \bar{Y}(\bar{t},\omega)$, one gets

$$\varphi_1(t, X(t,\omega), Y(t,\omega), a) = \varphi_1(0, X(0,\omega), Y(0,\omega), a) - \int_0^{\beta(t)} \bar{X}(s,\omega)\,\mathrm{d}\bar{B}_1(s),$$

$$\varphi_2(t, X(t,\omega), Y(t,\omega), a) = \varphi_2(0, X(0,\omega), Y(0,\omega), a) - \int_0^{\beta(t)} \bar{Y}(s,\omega)\,\mathrm{d}\bar{B}_1(s) + \int_0^{\beta(t)} \mathrm{d}\bar{B}_1(s) + \int_0^{\beta(t)} \mathrm{d}\bar{B}_2(s).$$

Because $\varphi_1(t, X(t,\omega), Y(t,\omega), a) = \bar{X}(\beta(t),\omega)$ and $\varphi_2(t, X(t,\omega), Y(t,\omega), a) = \bar{Y}(\beta(t), \omega)$, one has

$$\bar{X}(\beta(t),\omega) = \bar{X}(0,\omega) - \int_0^{\beta(t)} \bar{X}(s,\omega)\,\mathrm{d}s,$$

$$\bar{Y}(\beta(t),\omega) = \bar{Y}(0,\omega) - \int_0^{\beta(t)} \bar{Y}(s,\omega)\,\mathrm{d}s + \int_0^{\beta(t)} \mathrm{d}\bar{B}_1(s) + \int_0^{\beta(t)} \mathrm{d}\bar{B}_2(s).$$

This confirms that the Lie group of transformations (62) transforms any solution of equations (59) into a solution of the same equations in this particular case.

6. Conclusion

The definition of an admitted Lie group of transformations for stochastic differential equations was extended to stochastic differential equations with multi-dimensional Brownian motion. This approach includes dependent and independent variables in the transformation. The transformation of Brownian motion is defined by the transformation of dependent and independent variables. Correctness of all developed construction is strictly proven. Thus a correct approach for generalization of group analysis to stochastic differential equations has been developed. The developed theory was applied to a variety of stochastic differential equations. First, stochastic differential equations with one-dimensional Brownian motion were studied. Then the theory was extended to stochastic differential equations with multi-dimensional Brownian motion. For stochastic differential equations with one-dimensional Brownian motion, four applications were studied: a system describing the graph of Brownian motion, a system describing the Black and Scholes market, a system describing mean-reverting an Ornstein–Uhlenbeck process and a nonlinear Itô system. For stochastic differential equations with multi-dimensional Brownian motion, four applications were studied: a system describing location and motion, a system describing model for a vibrating string subject to a stochastic force, a system representing Ornstein–Uhlenbeck process and a nonlinear Itô system.

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